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Program Control of a Multichannel Object Search System with a Finite Number of Processing Lines in Each Channel

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Abstract—The problem of searching for observation objects appearing in accordance with the Poisson flow mathematical model is considered. This problem is solved in the case of a finite number of processing lines in the independent channels of a multichannel search system. Under a high-intensity flow of objects, this may cause queues in object processing. A set of queuing systems with independent incoming flows is selected as a model to determine a search law. The systems of Kolmogorov differential equations describe the dynamics of the probabilistic characteristics of their states under a finite number of processing lines in the search channels. Two formulations of the optimization problem are considered, namely, with probabilistic and time performance criteria. An iterative procedure is proposed to control the distribution of search intensities in the channels of the search system. Examples are given.

Keywords: search for observation objects, multichannel search system, distribution of search intensities, the system of Kolmogorov differential equations

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1. INTRODUCTION

The problems of rational effort distribution in search systems (SSs) [1–13] are an important class of problems in the general theory of observation control and management [14–17]. Their effective solution decreases the probability of undetected observation objects (OOs) in the SS coverage area, reduces the time of their search, improves the reliability of OO detection, etc. In particular, such problems include the search for and detection of space debris, since the corresponding flow of particles can be treated as a Poisson flow [18], car traffic monitoring on multilane highways, air traffic management near large airports, airspace use monitoring in megacities, etc.

The rules of solving the above class of problems can be applied to control other systems and queuing networks, including the management of water resources, power grids, data transmission processes in telecommunication systems, etc. [19–26].

The most general structural assumption regarding the mathematical model describing OOs in the search process is their arbitrary (in particular, unlimited) number and appearance in the field of view in accordance with a spatiotemporal random flow [27–30], usually Poisson.

In the latter case, the search problem was investigated in [6–10]. The common feature of the above works is the absence of constraints on the number of processing lines in the SS channels. Under certain conditions, this feature significantly simplified the mathematical models underlying the design of the distribution laws of search intensities. However, for real SSs with a finite number of processing lines in the channels, such distribution law design approaches cannot be used. This is due to the peculiarities of the mathematical models (the Kolmogorov equations) describing the dynamics of the probabilistic characteristics of the SS channel states in terms of a queuing problem with a finite number of processing lines in each channel [31–34].

In this regard, it is topical to control, via an appropriate law, the distribution of search intensities for multichannel parallel SSs with independent channels and a finite number of processing lines in each channel.

2. STRUCTURAL ANALYSIS OF THE MATHEMATICAL SEARCH MODEL

Let observation objects appear in the SS field of view X in accordance with a spatiotemporal Poisson flow φ [19–22]. Here, $X \subset R^n$ $(1 \leqslant n \leqslant 3)$, and $(x_1, \ldots, x_n)$ denotes the Cartesian frame in X.

Suppose that the field of view X is divided into I zones X_i, $i = \overline{1, I}$, each served by the corresponding ith SS channel.

By assumption, $X = \bigcup\limits_i X_i$, $X_i \bigcap X_j = \varnothing$, $i = \overline{1, I}$, $j = \overline{1, I}$, $i \neq j$. Then any two flows defined from φ as

$$\varphi_i(t) = \varphi(X_i, t), \quad \varphi_j(t) = \varphi(X_j, t), \quad i = \overline{1, I}, \quad j = \overline{1, I}, \quad i \neq j, \tag{2.1}$$

are Poisson and independent [27, 30].

According to [8, 9, 27, 34, 35], the intensity measures $\xi_i(t)$, $i = \overline{1, I}$, often called the rates of the temporal Poisson flows $\varphi_i(t)$, respectively, are known continuous and bounded deterministic functions. They characterize the temporal Poisson flows $\varphi_i(t)$, $i = \overline{1, I}$, as flows with variable parameters [27–29].

For each zone X_i, the probability of the appearance of one successive OO on a time interval $[t, t + \Delta t]$ is given by [32, 33]

$$\xi_i(t)\Delta t + o(\Delta t), \tag{2.2}$$

whereas the probability of the appearance of more than one OO constitutes $o(\Delta t)$. Here, $o(\Delta t)$ is the residual of an infinitesimal order higher than Δt.

Let us introduce the following constraint: for the ith SS channel, $i = \overline{1, I}$, the number of processing lines is finite and equal to a_i.

We define the search time interval as $\Omega = [0, \bar{t}\,]$. Be assumption, during this interval the ith SS channel provides a search intensity $\lambda_i(t) \geqslant 0$.

Then, if there are k OOs in the zone X_i, during the time $[t, t + \Delta t]$ at least one of them will be found with the probability [8–10, 32, 33]

$$\begin{aligned} &k\lambda_i(t)\Delta t + o(\Delta t), \quad k \leqslant a_i, \\ &a_i\lambda_i(t)\Delta t + o(\Delta t), \quad k > a_i. \end{aligned} \tag{2.3}$$

In other words, for $k > a_i$, the probability of detecting an OO on $[t, t + \Delta t]$ stops growing with increasing k.

The search intensities $\lambda_i(t)$ in the zones X_i, $i = \overline{1, I}$, are supposed to be unknown, nonnegative, continuous time-varying functions bounded on $[0, \bar{t}]$. They are to be determined by solving an optimization problem.

The search system problem in the zone X_i can be interpreted as servicing this zone with an incoming OO flow that forms a Poisson-type load with the intensity measure $\xi_i(t)$. As a rule, this class of problems is described using the mathematical apparatus of reproduction and death processes [31–33]. The dynamics of such processes are characterized using infinite-dimensional systems of Kolmogorov linear ordinary differential equations for the probabilities of process states. These systems form a set of Cauchy problems [32, 33, 36]

$$\begin{aligned}
&\dot{P}_{i0} = -\xi_i(t)P_{i0} + \lambda_i(t)P_{i1},\\
&\dot{P}_{ik} = -(\xi_i(t) + k\lambda_i(t))P_{ik} + \xi_i(t)P_{ik-1} + (k+1)\lambda_i(t)P_{ik+1}, \quad (k = 1, \ldots, a_i - 1),\\
&\dot{P}_{ik} = -(\xi_i(t) + a_i\lambda_i(t))P_{ik} + \xi_i(t)P_{ik-1} + a_i\lambda_i(t)P_{ik+1}, \quad k \geqslant a_i,\\
&P_{i0}(0) = 1, \quad P_{ik}(0) = 0, \quad i = \overline{1, I}, \quad t \in [0, \bar{t}],
\end{aligned} \tag{2.4}$$

where P_{i0} is the probability of absence of undetected OOs in X_i or the probability of an event associated with the non-engagement of all processing lines of the ith SS channel for $t \in \Omega$; P_{ik} is the probability of presence of undetected OOs in X_i k or the probability of an event associated with the engagement of k processing lines of the ith SS channel if $1 \leqslant k < a_i$, or a_i processing lines if $k \geqslant a_i$; the functions $\{\lambda_i(t),\ \xi_i(t),\ i = \overline{1, I}\}$ on the interval Ω are supposed to be non-negative, continuous, and bounded.

The Cauchy problem (2.4) reflects the physical meaning of the search carried out by independent channels of the SS with a finite number of processing lines, sequentially in time as the OOs appear in the zones X_i, $i = \overline{1, I}$.

When determining a control law for the distribution of search intensities in the SS, it is difficult to use the mathematical models (2.4) due to their infinite dimension. In addition, the structures of these models eliminate the possibility of their convolution, e.g., in terms of the expected (mean) number of undetected OOs in each SS channel, as was done in [10].

By the service elements (servers) of the ith channel we will understand its processing lines. Then there is no queue in search information processing in the ith channel if:

— None of its servers is busy or none of the processing lines is engaged in processing, which corresponds to $k = 0$.

— k servers ($k = 1, \ldots, a_i$) are busy or k processing lines are engaged in processing.

The above states of the ith channel can be represented by the set

$$S_i = \left\{ s_{i0}, s_{i1}, \ldots, s_{ik}\big|_{k=a_i} \right\}, \quad i = \overline{1, I}, \tag{2.5}$$

where each subset s_{ik} ($k = 0, 2, \ldots, a_i$) includes $C_{a_i}^k = \frac{a_i!}{k!(a_i-k)!}$ elements.

3. PROBLEM STATEMENT

Let p_i, $i = \overline{1, I}$, denote the probability of an event associated with a single exit of the ith SS channel from the set (2.5) given $0 \leqslant k \leqslant a_i$ busy servers at the initial time instant. Note that "single exit" means that the channel leaves S_i exactly once.

Such events are joint and independent, and the probability of their sum is given by the relation [37]

$$p_\Sigma(t) = p_1(t) + p_2(t)(1 - p_1(t)) + \ldots + p_I(t)\prod_{i=1}^{I-1}(1 - p_i(t)). \tag{3.1}$$

We define the first performance criterion for the search system as

$$\Upsilon_1 = p_\Sigma(\bar{t}) + \eta\int\limits_0^{\bar{t}} \lambda(t)^{\mathrm{T}}\lambda(t)dt \to \min_{\lambda\in\Lambda}, \tag{3.2}$$

where $\lambda(t) = [\lambda_1(t) \ldots \lambda_I(t)]^{\mathrm{T}}$ is the vector of search intensities; $\eta \in R^1$, $\eta > 0$ is a weight coefficient; Λ is a set of admissible control laws (distribution laws of search intensities).

The set of admissible control laws consists of nonnegative and continuous time-varying functions bounded on $[0, \bar{t}]$.

The criterion (3.2) is probabilistic. It assumes minimizing the sum of two components. The first component is the probability with which the state of at least one SS channel will exit the set (2.5) at the terminal time instant $\bar{t}$ of observations. The second component is an analog of the search energy cost of the SS.

Consider the second performance criterion. To do this, we introduce the mean dwell times of the SS channels in the states excluding any queues in search information processing: $m_i = \mathrm{M}[\tau_i]$, $i = \overline{1, I}$, where τ_i is a random variable specifying the dwell time of the ith SS channel in the state set (2.5). We define the second performance criterion as

$$\Upsilon_2 = I\bar{t} - \sum_{i=1}^{I} m_i + \eta \int_0^{\bar{t}} \lambda(t)^{\mathrm{T}} \lambda(t) dt \to \min_{\lambda \in \Lambda}. \tag{3.3}$$

The criterion (3.3) is a time criterion. It assumes minimizing the sum of two components at the terminal time instant of observations. The first component is the sum of the mean dwell times of the SS channels in the state sets (2.5), taken with a minus sign. The second component, as in (3.2), is an analog of the search energy cost of the SS.

The problem is to design control laws for the distributions of search intensities $\lambda_i(t)$, $i = \overline{1, I}$, $t \in \Omega$, in (2.4) between the channels of the multichannel SS with a finite number of processing lines in each channel that serve the non-intersecting zones X_i, $i = \overline{1, I}$, of the SS field of view X via the probabilistic (3.2) and time (3.3) performance criteria.

4. THE DISTRIBUTIONS OF SEARCH INTENSITIES IN THE MULTICHANNEL SS WITH A FINITE NUMBER OF PROCESSING LINES: CONTROL DESIGN VIA THE PROBABILISTIC CRITERION

Consider the control design procedure for the search for observation objects from the spatiotemporal Poisson flow φ in the multichannel search system with independent channels and a finite number of processing lines in each channel in terms the probabilistic performance criterion (3.2). To do this, we establish the following result.

Proposition 1. *The probability of an event associated with a single exit of the ith SS channel $(i = \overline{1, I})$ from the state set S_i (2.5) provided the belonging of its state to this set at $t = 0$ is given by*

$$p_i(t) = \bar{P}_{i\ a_i+1}(t), \tag{4.1}$$

where the probability $\bar{P}_{i\ a_i+1}(t)$ is obtained by solving the system of differential equations

$$\begin{aligned}
&\dot{\bar{P}}_{i0}(t) = -\xi_i(t)\bar{P}_{i0}(t) + \lambda_i(t)\bar{P}_{i1}(t),\\
&\dot{\bar{P}}_{ik}(t) = -(\xi_i(t) + k\lambda_i(t))\bar{P}_{ik}(t) + \xi_i(t)\bar{P}_{ik-1}(t) + (k+1)\lambda_i(t)\bar{P}_{ik+1}(t), \quad (k = 1, \ldots, a_i - 1),\\
&\dot{\bar{P}}_{ia_i}(t) = -(\xi_i(t) + a_i\lambda_i(t))\bar{P}_{ia_i}(t) + \xi_i(t)\bar{P}_{ia_i-1}(t),\\
&\dot{\bar{P}}_{ia_i+1}(t) = \xi_i(t)\bar{P}_{a_i}(t), \quad i = \overline{1, I},
\end{aligned} \tag{4.2}$$

with the initial conditions

$$\sum_{k=0}^{a_i} \bar{P}_{ik}(0) = 1, \quad \bar{P}_{i\ a_i+1}(0) = 0. \tag{4.3}$$

The proof is postponed to the Appendix.

Note that equations (4.2) form a set of fictitious dynamic systems [15] with vector control $\lambda = [\lambda_1\ \lambda_2\ \ldots\ \lambda_I]^{\mathrm{T}}$. The initial conditions (4.3) determine the belonging of the states of the ith channel to the set S_i at $t = 0$ and correspond to the absence of a queue at its inputs.

Thus, for (3.1) we have

$$p_\Sigma(\bar{t}) = \bar{P}_{1\ a_1+1}(\bar{t}) + \bar{P}_{2\ a_2+1}(\bar{t})(1 - \bar{P}_{1\ a_1+1}(\bar{t})) + \ldots + \bar{P}_{I\ a_I+1}(\bar{t}) \prod_{i=1}^{I-1}(1 - \bar{P}_{i\ a_i+1}(\bar{t})). \tag{4.4}$$

In view of (3.2) and (4.2), we define the Hamiltonian

$$H = \sum_{i=1}^{I} \psi_i^{\mathrm{T}} A_i \mathrm{P}_i + \eta\lambda^{\mathrm{T}}\lambda, \tag{4.5}$$

where $\psi_i = \psi_i(t) \in R^{a_i+2}$ is the vector of conjugate variables; $\mathrm{P}_i(t) = \left[\bar{P}_{i0}(t)\ \bar{P}_{i1}(t)\ \ldots\ \bar{P}_{i\ a_i+1}(t)\right]^{\mathrm{T}}$; $A_i \in R^{(a_i+2)\times(a_i+2)}$, $A_i = \xi_i B_{i1} + \lambda_i B_{i2}$; the structure of the matrices B_{i1} and B_{i2} follows from (4.2) and is presented in the Appendix.

A peculiarity of the matrices B_{i1} and B_{i2} is the zero sum of each column.

From (4.5) we obtain the systems of equations for $\psi_i(t)$, $i = \overline{1, I}$:

$$\dot{\psi}_i = -\frac{\partial}{\partial \mathrm{P}_i}\mathrm{H} = -\left(\xi_i B_{i1}^{\mathrm{T}} + \lambda_i B_{i2}^{\mathrm{T}}\right)\psi_i, \quad t \in \Omega. \tag{4.6}$$

Due to (4.4), the boundary value conditions for (4.6) can be written as

$$\psi_i(\bar{t}) = \frac{\partial}{\partial \mathrm{P}_i(\bar{t})} p_\Sigma(\bar{t}) = \left[0 \quad 0 \quad \ldots \quad \prod_{j=1, j\neq i}^{I}(1 - \bar{P}_{j a_j}(\bar{t}))\right]^{\mathrm{T}}, \quad i = \overline{1, I}. \tag{4.7}$$

The minimum condition of (4.5) in λ yields

$$\frac{\partial}{\partial \lambda_i}\mathrm{H} = \psi_i^{\mathrm{T}} B_{i2}\mathrm{P}_i + 2\eta\lambda_i = 0, \quad i = \overline{1, I}. \tag{4.8}$$

Considering (4.8), we define the optimal control

$$\lambda_i = \frac{-\psi_i^{\mathrm{T}} B_{i2} P_i}{2\eta}, \quad i = \overline{1, I}. \tag{4.9}$$

To solve the two-point boundary value problem (4.2), (4.6), (4.7), we will apply the Krylov–Chernous'ko method of successive approximations [14, 15, 39]. Assume that $\bar{P}_{i0}(0) = 1$ to specify (4.3).

Let a search control law, i.e., a search intensity vector λ^q, be formed at the qth step of the iterative procedure.

The method of successive approximations involves the following operations.

1. Equations (4.2) are solved on the time interval Ω forward, and the vector of variables P_i^q, $i = \overline{1, I}$, corresponding to the distribution of search intensities λ_i^q, $i = \overline{1, I}$, is determined.

2. Based on these solutions, the terminal conditions for the conjugate variables $\psi_i^q(\bar{t})$, $i = \overline{1, I}$, are formed according to (4.7).

3. The system of equations (4.6) is solved on the time interval Ω backward with the terminal conditions (4.7), and the vector of conjugate variables $\psi_i^q(t)$, $t \in \Omega$, $i = \overline{1, I}$, corresponding to the control laws λ_i^q, $i = \overline{1, I}$, and the vector of variables P_i^q, $i = \overline{1, I}$, is determined.

4. Based on the vectors $\mathrm{P}_i^q(t)$ and $\psi_i^q(t)$, $i=\overline{1,I}$, the intermediate control vector corresponding to the $(q+1)$th step of the iterative procedure is calculated according to (4.9):

$$\tilde{\lambda}_i^{q+1}=\frac{-(\psi_i^q)^{\mathrm{T}}B_{i2}P_i^q}{2\eta},\quad i=\overline{1,I},\quad t\in\Omega. \tag{4.10}$$

5. By the principle of partial control update [5, 8, 28], the search intensity control laws for the $(q+1)$th iteration are calculated using the distribution of search intensities λ_i^q, $i=\overline{1,I}$, obtained at the previous step:

$$\lambda_i^{q+1}(t)=\varepsilon^q\tilde{\lambda}_i^{q+1}(t)+(1-\varepsilon^q)\lambda_i^q(t),\quad i=\overline{1,I},\quad t\in\Omega, \tag{4.11}$$

where $\varepsilon^q\in(0,1)$.

The parameter ε^q denotes the degree of updating of the search control laws $\lambda_i^q(t)$, $i=\overline{1,I}$, $t\in\Omega$. It is determined by minimizing the objective function of the performance criterion (3.2) at the corresponding step of the iterative procedure.

Next, the control law $\lambda_i^{q+1}(t)$ is used as the initial one for the $(q+2)$th step, and operations 1–5 of the iterative procedure are repeated.

Note that for the initial step of the iterative procedure ($q=0$), the initial distribution of search intensities λ_i^0, $i=\overline{1,I}$, is selected from the set Λ of admissible control laws.

The optimal control (i.e., the optimal distribution law of search intensities) is given by

$$\lambda_{i\mathrm{OS}}(t)=\lim_{q\to\infty}\lambda_i^q(t),\quad i=\overline{1,I},\quad t\in\Omega. \tag{4.12}$$

At each step of the iterative procedure, the partial control variation (4.11) obtained by solving the optimization problem is intended to reduce the values of the objective functional. The values $\varepsilon^q\in(0,1)$ in (4.11) are chosen from the condition of its maximum decrease. On the other hand, according to (3.2), the objective functional corresponding to this criterion is bounded from below. (At least, the functional under consideration cannot be negative.) Thus, the limit (4.12) exists.

In practice, one often performs $q\leqslant Q$ (finitely many) iterations, where Q is the step number after which the variations of the objective functional of the performance criterion (3.2) become insignificant. By assumption, $\lambda_{i\mathrm{OS}}(t)\simeq\lambda_i^Q(t)$, $i=\overline{1,I}$.

5. THE DISTRIBUTIONS OF SEARCH INTENSITIES IN THE MULTICHANNEL SS WITH A FINITE NUMBER OF PROCESSING LINES: CONTROL DESIGN VIA THE TIME CRITERION

Let the upper limit $\bar{t}$ of the observation interval Ω be chosen so that the densities $w_i(t)$, $i=\overline{1,I}$, of the dwell times τ_i of the SS channels in the corresponding sets (2.5) satisfy the conditions

$$\int_0^{\bar{t}} w_i(t)dt=1,\ \forall i,i=\overline{1,I}. \tag{5.1}$$

In other words, these densities are finite on Ω.

In view of (5.1), we transform the performance criterion (3.3) as follows.

Assertion 1. *Under condition* (5.1)*, the mean dwell time τ_i of the ith SS channel in the state set S_i* (2.5) *with the initial conditions* (4.3) *(a single stay) is given by*

$$m_i=\int_0^{\bar{t}}(1-\bar{P}_{ia_i+1}(t))dt. \tag{5.2}$$

The proof of this result is provided in the Appendix.

Considering (5.2), we transform the performance criterion (3.3) to

$$\Upsilon_2 = \int\limits_0^{\bar{t}} \left[\eta\lambda(t)^{\mathrm{T}}\lambda(t) + \sum_{i=1}^{I} \bar{P}_{ia_i+1}(t) \right] dt \to \min_{\lambda\in\Lambda}. \tag{5.3}$$

The mathematical models (4.2) and (5.3) are associated with the Hamiltonian

$$H = \sum_{i=1}^{I} \left(\psi_i^{\mathrm{T}} A_i \mathrm{P}_i + \bar{P}_{ia_i+1}(t) \right) + \eta\lambda^{\mathrm{T}}\lambda. \tag{5.4}$$

The equations for the conjugate variables have the form

$$\dot{\psi}_i = -\frac{\partial}{\partial \mathrm{P}_i}\mathrm{H} = -(\xi_i B_{i1}^{\mathrm{T}} + \lambda_i B_{i2}^{\mathrm{T}})\psi_i - U_i, \quad i = \overline{1, I}, \quad t \in \Omega, \tag{5.5}$$

where the matrices B_{i1} and B_{i2}, as in the first case, are determined according to (A.2); $U_i \in R^{a_i+2}$, $U_i = [0\ 0\ \ldots\ 0\ 1]^{\mathrm{T}}$.

Due to (5.3), the boundary value conditions for (5.5) take the form $\psi_i(\bar{t}) = 0$, $i = \overline{1, I}$; by (5.4), the structure of the resulting control laws coincides with (4.9).

Like for the probabilistic criterion, the two-point boundary value problem (4.2), (5.5) is solved using the Krylov–Chernous'ko successive approximation procedure. It is applied similarly to the previous section.

Owing to the structure of mathematical models in the case under consideration, the computational optimization procedure can be decomposed channel-wise.

If the length of the observation interval is inconsistent with the structure of at least one of the densities $w_i(t)$, $i = \overline{1, I}$ (i.e., equality (5.1) holds only approximately), the solution becomes suboptimal.

6. THE DISTRIBUTIONS OF SEARCH INTENSITIES IN THE MULTICHANNEL SS WITH A FINITE NUMBER OF PROCESSING LINES: CONTROL DESIGN EXAMPLES

Example 1. Consider a two-channel SS ($I = 2$) with three and four processing lines in the first and second channels, respectively. The fictitious dynamic systems corresponding to the SS are described by a mathematical model consisting of two systems of equations (4.2) with $a_1 = 3$ and $a_2 = 4$, respectively. In this case, the performance criterion (3.2) takes the form

$$\Upsilon_1 = \bar{P}_{14}(\bar{t}) + \bar{P}_{25}(\bar{t}) - \bar{P}_{14}(\bar{t})\bar{P}_{25}(\bar{t}) + \eta \int\limits_0^{\bar{t}} \left(\lambda_1^2(t) + \lambda_2^2(t) \right) dt \to \min_{\lambda_1,\lambda_2\in\Lambda}. \tag{6.1}$$

The direct systems of equations (4.2) for $i = 1, 2$ were solved with the initial conditions corresponding to (4.3),

$$\begin{aligned}
&\bar{P}_{10}(0) = \bar{P}_{20}(0) = 1,\\
&\bar{P}_{11}(0) = \bar{P}_{12}(0) = \bar{P}_{13}(0) = \bar{P}_{14}(0) = 0,\\
&\bar{P}_{21}(0) = \bar{P}_{22}(0) = \bar{P}_{23}(0) = \bar{P}_{24}(0) = \bar{P}_{25}(0) = 0
\end{aligned}$$

on the search time interval $\Omega = [0, 5]$. From this point onwards, the parameters are presented in dimensionless units.

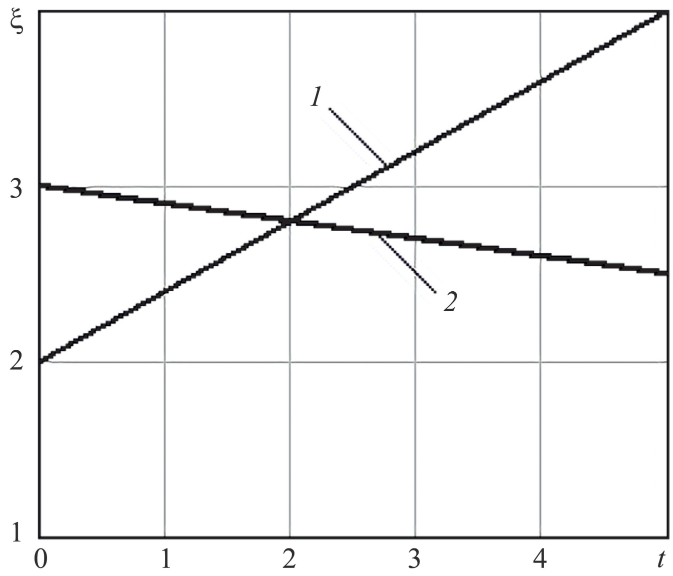

Fig. 1.

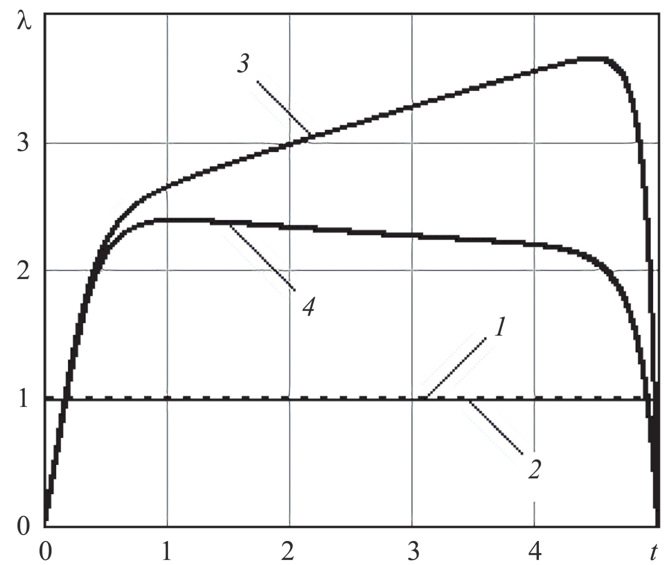

Fig. 2.

The conjugate systems (4.6) are completely determined by the structures of the matrices (A.2) with

$$B_{11}, B_{12} \in R^{5\times 5} \quad \text{and} \quad B_{21}, B_{22} \in R^{6\times 6}. \tag{6.2}$$

They were solved in reverse time (backward) with the terminal conditions corresponding to (4.7):

$$\begin{gathered} \psi_1(\bar{t}) = \begin{bmatrix} 0 & 0 & 0 & 0 & [1 - \bar{P}_{25}(\bar{t})] \end{bmatrix}^{\mathrm{T}}, \\ \psi_2(\bar{t}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & [1 - \bar{P}_{14}(\bar{t})] \end{bmatrix}^{\mathrm{T}}. \end{gathered} \tag{6.3}$$

The appearance intensities of OOs in the zones X_1 and X_2 of the SS search area were supposed to be linear functions of time: $\xi_1(t) = 2 + 0.4t$ and $\xi_2(t) = 3 - 0.1t$ (lines *1* and *2* in Fig. 1).

The initial distributions of search intensities λ_1^0 and λ_2^0 were set uniform over the observation interval (lines *1* and *2* in Fig. 2). The equations were solved on the search time interval with a step of $\Delta = 10^{-3}$.

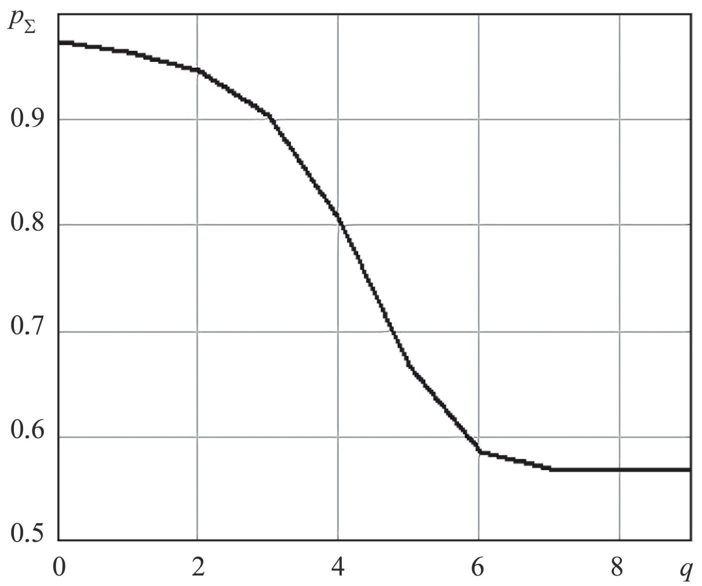

Fig. 3.

Figure 3 shows the probability $p_\Sigma(\bar{t})$ of exiting the state sets

$$S_1 = \{s_{10},\ s_{11},\ \ldots,\ s_{13}\}, \quad S_2 = \{s_{20},\ s_{21},\ \ldots,\ s_{24}\} \tag{6.4}$$

by at least one of the two SS channels depending on the iteration number q in the Krylov–Chernous'ko successive approximation procedure.

The variations in the probability component of the objective functional of the performance criterion (6.1) become insignificant already for $q \geqslant Q = 7$. The structure of the optimal control laws for the distribution of search intensities is presented in Fig. 2: curves *3* and *4* for the first and second channels, respectively.

Note that these dependencies are consistent with the variation in the appearance intensities of OOs from the flow (Fig. 1) in the corresponding SS channels.

The relative gain from optimization via the probabilistic criterion makes up $\delta = \frac{p_{\Sigma 0}(\bar{t}) - p_{\Sigma \mathrm{OS}}(\bar{t})}{p_{\Sigma 0}(\bar{t})} \simeq 0.42$, where $p_{\Sigma 0}(\bar{t})$ is the probability of exiting the state sets (6.4) by at least one of the two SS channels under λ_1^0, λ_2^0; $p_{\Sigma \mathrm{OS}}(\bar{t})$ is the same probability under $\lambda_i^Q(t) \simeq \lambda_{i\mathrm{OS}}(t)$, $i = 1, 2$.

Example 2. For the two-channel search system of Example 1, consider the control design problem for the distributions of search intensities via the time performance criterion (3.3) in the form (5.3). In this case, the criterion becomes

$$\Upsilon_2 = \int\limits_0^{\bar{t}} \Big[\eta(\lambda_1^2(t) + \lambda_2^2(t)) + \bar{P}_{14}(t) + \bar{P}_{25}(t)\Big]\, dt \to \min_{\lambda_1, \lambda_2 \in \Lambda}. \tag{6.5}$$

The direct systems of equations remain unchanged and have the form (4.2). The systems of equations for the conjugate variables correspond to (5.5) with the matrices (6.2), the vectors $U_1 \in R^5$ and $U_2 \in R^6$, and the zero terminal conditions $\psi_i(\bar{t}) = 0$, $i = 1, 2$.

The appearance intensities of OOs in the zones X_1 and X_2 were supposed to be linear functions of time: $\xi_1(t) = 9 + 0.5t$ and $\xi_2(t) = 12 - 0.1t$ (lines *1* and *2*, respectively, in Fig. 4).

The initial distributions of search intensities λ_1^0 and λ_2^0 were set uniform over the observation interval (lines *1* and *2* in Fig. 5).

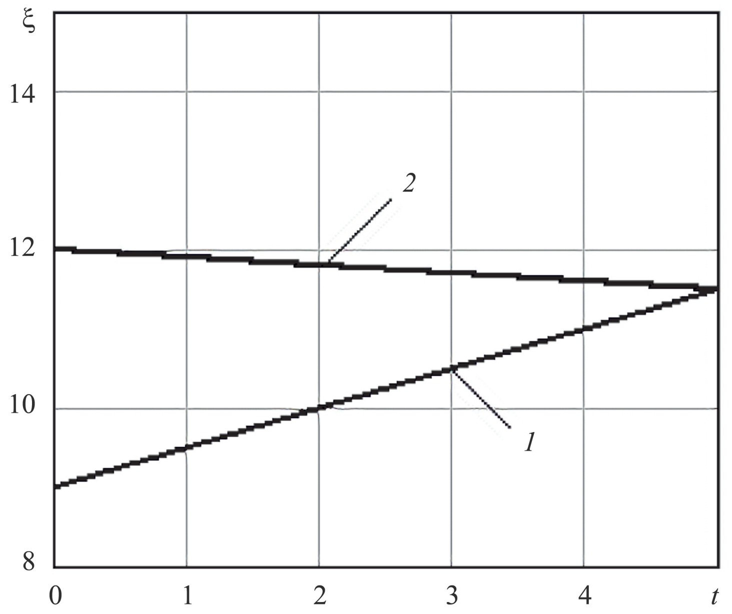

Fig. 4.

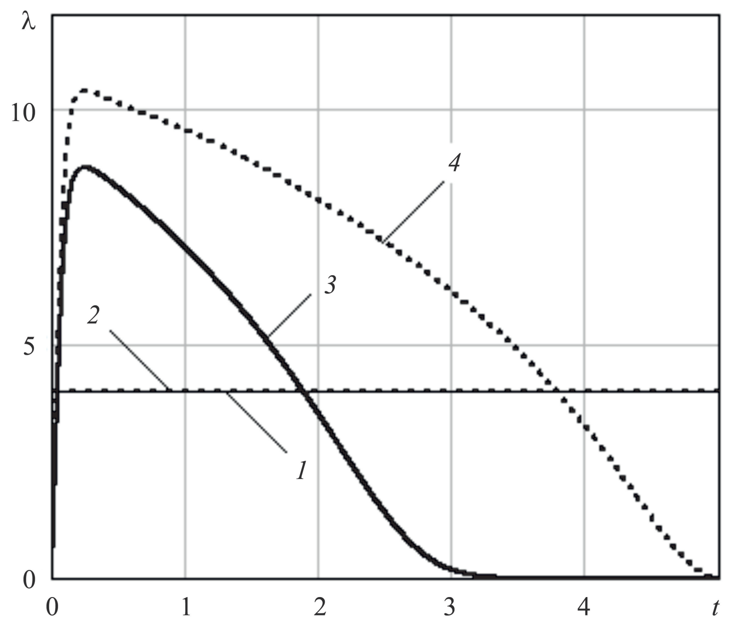

Fig. 5.

Figure 6 shows the mean dwell times m_1 (curve *1*) and m_2 (curve *2*) of the SS channels are in the state sets (6.4) and their sum (curve *3*) depending on the iteration number q in the Krylov–Chernous'ko successive approximation procedure. The time characteristics had insignificant changes for $q \geqslant Q = 9$.

The smaller growth of m_1 compared to m_2 is due to the smaller number of states in the set S_1 compared to the set S_2.

The optimal control laws for the distribution of search intensities are demonstrated in Fig. 5: curves *3* and *4* for the first and second SS channels, respectively. The lower search intensity in the first channel is due to the lower appearance intensity of OOs in it (Fig. 4).

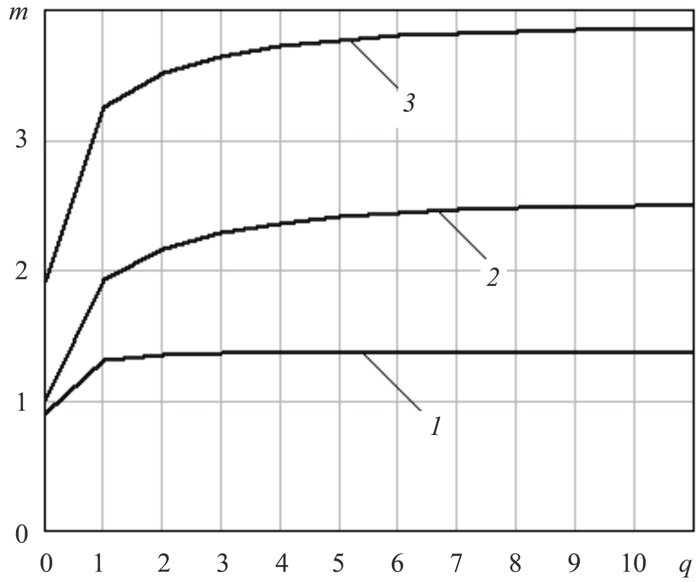

Fig. 6.

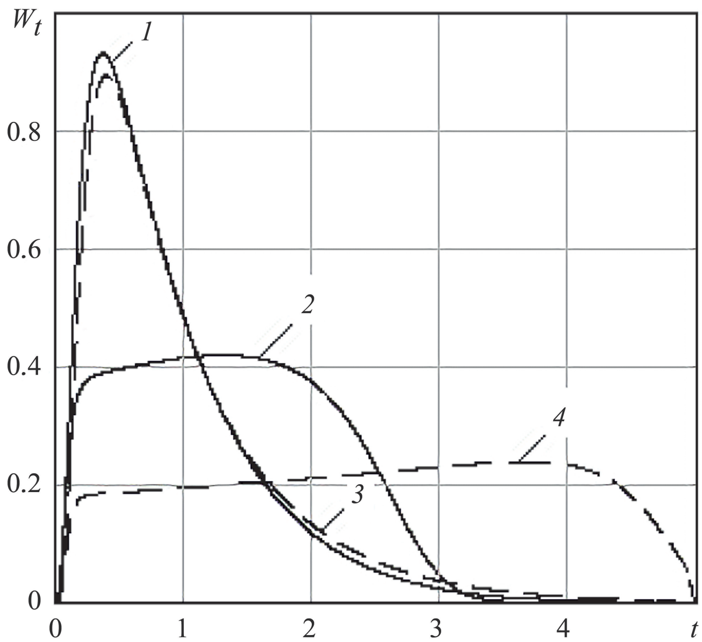

Fig. 7.

Figure 7 presents the structure of the densities of the dwell times of the SS channels in the state sets (6.4): for $t = 0$, curves *1* and *3* for the first and second channels, respectively; for $t = 5$, curves *2* and *4* similarly.

In this example, the optimization procedure increased the total mean dwell time of the SS channels in the state sets (6.4), excluding any queues in search information processing, by more than 2 times.

Considering the structure of the above mathematical models, it is easy to establish the following fact: under the hypotheses of Proposition 1, the search intensities obtained by solving the optimization problems for both performance criteria ((3.2) and and (3.3)) always belong to the class of nonnegative, continuous, and bounded functions.

7. CONCLUSIONS

The above control design approach to the search for OOs from a spatiotemporal Poisson flow in a multichannel SS is oriented toward several practically important cases with a finite number of processing lines in each SS channel. In such cases, it is impossible to apply methods associated with the convolution of an infinite system of Kolmogorov equations, e.g., in terms of the mean number of undetected OOs, particularly considered in [10].

The approach is based on the transition from infinite-dimensional systems of differential equations describing the probabilistic characteristics of SS channel states to finite-dimensional auxiliary systems. Their dimensions are determined by the number of processing lines in each channel.

The parameters of the auxiliary systems of equations are the probabilistic characteristics of dwell of the processing lines of SS channels in the state sets excluding any queues for search information processing. With these parameters, it is possible to design control laws for the distribution of search intensities from two viewpoints: minimizing the probability of exiting the above state sets by at least one SS channel or maximizing the mean dwell time of the channels in these sets.

The solution of the first optimization problem with the probabilistic performance criterion (3.2) is more complex due to the structure of the terminal component of this criterion. Characterized by the total dimension of all auxiliary systems of equations (4.2), the dimension of this component is given by $\sum_{i=1}^{I} a_i + 2I$.

In the second case, the general control design procedure for search intensities can be decomposed channel-wise.

Due to a rather high level of complexity, it is advisable to solve the arising optimization problems using the Krylov–Chernous'ko method of successive approximations together with the principle of partial control update. The examples have shown the possibility of their effective solution using the above approach, particularly the gain from optimization.

Note that the approach can be generalized to the case when the search intensities of OOs in different SS channels are not independent and satisfy the constraint $\sum_{i=1}^{I} \lambda_i(t) = \lambda_\Sigma = \text{const}$.

APPENDIX

Proof of Proposition 1

We associated with the system of equations (2.4) the state graph [38] presented in Fig. 8.

The graph reflects the sequence of variations in the states of the ith channel with a finite number of processing lines. We select on this graph the states corresponding to the set

$$S_i = \left\{ s_{i0},\ s_{i1},\ \ldots,\ s_{ik}\big|_{k=a_i} \right\} \to \left\{ \bar{s}_{i0},\ \bar{s}_{i1}, b\ \ldots,\ \bar{s}_{ik}\big|_{k=a_i} \right\} = \bar{S}_i. \tag{A.1}$$

For (A.1), we construct the auxiliary state graph (Fig. 9) as a fragment of the original graph (Fig. 8) with only one absorbing state $\bar{s}_{ia_i+1}$ added.

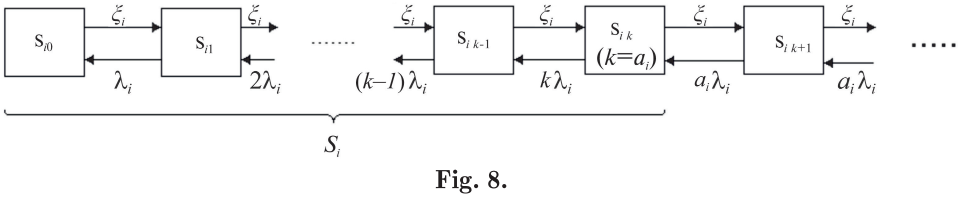

Fig. 8.

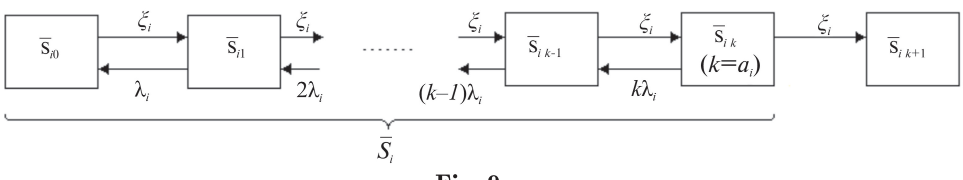

Fig. 9.

It can be shown [38] that the time variation of the probabilistic characteristics $\bar{P}_{i0}(t)$, $\bar{P}_{i1}(t), \ldots, \bar{P}_{ia_i}(t), \bar{P}_{ia_i+1}(t)$ of the states $\left\{\bar{s}_{i0}, \bar{s}_{i1}, \ldots, \bar{s}_{ik}\big|_{k=a_i}, \bar{s}_{ia_i+1}\right\}$ corresponding to the graph in Fig. 9 is described by the system of differential equations (4.2). For the initial conditions (4.3), the probability of an event associated with a single exit from the set $\bar{S}_i$ (and, accordingly, from the set S_i) is determined by the probability of passing from the state $\bar{s}_{ik}\big|_{k=a_i}$ to $\bar{s}_{i\,a_i+1}$. In other words, it corresponds to the component of the solution of system (4.2) determined by the probability $\bar{P}_{i\,a_i+1}(t)$.

The proof of Proposition 1 is complete.

The structure of the matrices B_{i1}, $B_{i2} \in R^{(a_i+2)\times(a_i+2)}$

$$B_{i1} = \begin{bmatrix} -1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & -1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & -1 & \ldots & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & \ldots & -1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}, \quad B_{i2} = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 2 & 0 & \ldots & 0 & 0 \\ 0 & 0 & -2 & 3 & \ldots & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & \ldots & a_i & 0 \\ 0 & 0 & 0 & 0 & \ldots & -a_i & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}. \tag{A.2}$$

Proof of Proposition 2

We write the last equation of the auxiliary system of equations (4.2) corresponding to the ith channel:

$$\dot{\bar{P}}_{ia_i+1}(t) = \xi_i(t)\bar{P}_{ia_i}(t), \quad \bar{P}_{ia_i+1}(0) = 0. \tag{A.3}$$

Recall that the parameter $\bar{P}_{ia_i+1}(t)$ means the probability of an event associated with a single exit of the ith SS channel from the state set S_i (2.5). Therefore, its derivative (A.3) determines the density $w_i(t)$ of the random dwell time τ_i of this channel in the state set S_i. In other words [38],

$$w_i(t) = \xi_i(t)\bar{P}_{ia_i}(t). \tag{A.4}$$

Now we multiply the left- and right-hand sides of (A.3) by t and integrate over the observation interval Ω. As a result, in view of (A.4), we obtain

$$\int\limits_0^{\bar{t}} t\dot{\bar{P}}_{ia_i+1}(t)dt = \int\limits_0^{\bar{t}} t\xi_i(t)\bar{P}_{ia_i}(t)dt = \int\limits_0^{\bar{t}} tw_i(t)dt. \tag{A.5}$$

From (A.5) it follows that

$$\int\limits_0^{\bar{t}} t\dot{\bar{P}}_{ia_i+1}(t)dt = m_i, \tag{A.6}$$

where $m_i = \int_0^{\bar{t}} tw_i(t)dt$ is the mean of the random dwell time τ_i.

Integrating by parts the left-hand side of (A.6) yields

$$t\bar{P}_{ia_i+1}(t)\Big|_0^{\bar{t}} - \int\limits_0^{\bar{t}} \bar{P}_{ia_i+1}(t)dt = m_i. \tag{A.7}$$

Considering condition (5.1), we finally arrive at

$$\int_0^{\bar{t}} (1 - \bar{P}_{ia_i+1}(t))dt = m_i. \tag{A.8}$$

The proof of Proposition 2 is complete.

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